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AN EXTENSION OF WEYL'S LEMMA TO INFINITE DIMENSIONS(1)

BY
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ABSTRACT. A theory of distributions analogous to Schwartz distribution theory is formulated for separable Banach spaces, using abstract Wiener space techniques. A distribution T is harmonic on an open set U if for any test function f on U, $T(\Delta f) = 0$, where Δf is the generalized Laplacian of f. We prove that a harmonic distribution on U can be represented as a unique measure on any subset of U which is a positive distance from U^C. In the case where the space is finite dimensional, it follows from Weyl's lemma that the measure is in fact represented by a C^∞ function. This functional representation cannot be expected in infinite dimensions, but it is shown that the measure has smoothness properties analogous to infinite differentiability of functions.

1. **Introduction.** The distribution theory formulation of Weyl's lemma, due to L. Schwartz, states that if T is a harmonic distribution on a domain in $\mathbf{R}^n$, then T is a C^∞ function. In this paper we develop a theory of distributions on a separable real Banach space, for which an analog of Weyl's lemma holds.

In general, there is no Lebesgue measure available on an arbitrary separable Banach space. However Gross has developed a measure theoretic structure well-suited to the study of potential theory on these spaces [5], [6]. Since Weyl's lemma is naturally related to potential theory, it is appropriate to use this structure in our extension. The definitions of test functions and distributions are developed in this context, using the Schwartz theory as a model as much as possible. Thus, test functions are infinitely differentiable in a weak sense related to the abstract Wiener structure and they have bounded support. The lack of compactness in the supports is partially compensated for by requiring Lip 1 continuity of the function and its derivatives. Test functions converge if all their derivatives converge pointwise and boundedly, with appropriate restrictions on their supports. Distributions are elements of the corresponding topological dual space.

The main difference between Weyl's lemma on finite dimensional spaces and its extension to separable Banach spaces is that on the latter, a harmonic distribution is a measure but is not necessarily representable as a function. The Green measure G_x offers an example of this difference. On $\mathbf{R}^n - \{x\}$, G_x is a

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harmonic distribution, represented by the functions $1/\|y - x\|^{n-2}$. On a separable Banach space equipped with an abstract Wiener structure, there is a measure with many of the potential theoretic properties of a Green measure. We shall show that this measure is a harmonic distribution on $B - \{x\}$. On an infinite dimensional space, it cannot be represented as a function.

This difference between the finite and infinite dimensional versions is a consequence of the lack of Lebesgue measure rather than a loss of regularity. In fact the measure representing the distribution has the following differentiability property: when convolved with a bounded Lip 1 function whose support satisfies certain conditions, the resulting function is infinitely differentiable (in the weak sense mentioned above) in a neighborhood of 0.

Our main result is achieved in three steps: first we prove that any distribution can be approximated locally by a sequence of measures which are harmonic when the distribution is. We show that for a large class of functions f, there are associated test functions ϕ_f, which are modifications of the Laplacian of the potential of f, for which $\int f d\mu = \int \phi_f d\mu$ when μ is a harmonic measure. It is shown that this representation holds as the measures converge to the distribution and from this representation of the distribution we conclude that it is locally a measure.

To establish some of the notation of this paper, we list a few pertinent results from potential theory on abstract Wiener spaces. The following summary is not self-contained and the reader is referred to [5], [6].

An *abstract Wiener space* consists of a real, separable Hilbert space H, a Banach space B, and a family of probability measures $\{p_t\}_{t \geq 0}$. B is the completion of H with respect to a *measurable* norm $\|\cdot\|$. A measurable norm is one for which, given $\varepsilon > 0$, there is a finite dimensional orthogonal projection P_0 on H with the following property: if P is a finite dimensional projection orthogonal to P_0 and m is a Gaussian measure with mean zero and variance one on PH, then $m\{h\colon \|Ph\| > \varepsilon\} < \varepsilon$. Intuitively, the unit ball of a measurable norm contains an arbitrarily large portion of infinitely many coordinate axes. The measures $\{p_t\}_{t \geq 0}$ are characterized by the following Gaussian property: if $e_1, \ldots, e_n$ are elements of B^* whose images under the dual injection $B^* \subset H^*$ are orthonormal, and A is a Borel set in R^n, then

$$p_t\{x \in B\colon (\langle e_1, x\rangle, \ldots, \langle e_n, x\rangle) \in A\} = \left(\frac{1}{2\pi t}\right)^{n/2} \int_A e^{-|u|^2/2t}\, du.$$

Given a separable real Hilbert space H, one can always construct an abstract Wiener space with H as the first element and given a separable Banach space B, one can always construct an abstract Wiener space with B as the second element. In both cases the resulting abstract Wiener spaces are not unique. If one begins with H as the first element, then all resulting abstract Wiener spaces are isomorphic for the purposes of integration theory. This is not true if one begins with the second element B. In this case, for each abstract Wiener structure (i.e.

for each H properly embedded in B) there is a corresponding set of test functions and distributions. This occurs because the differentiability of test functions is determined by smoothness on H-cosets.

A probabilistic interpretation of an abstract Wiener space is that the family $\{p_t(x, \cdot)\}_{t \geq 0}$, defined by $p_t(x, A) = p_t(A - x)$, are the transition functions for a Markov process $(\Omega, \mathcal{F}, \{P_x\}_{x \in B}, \{X_t\}_{t \geq 0})$ with state space B and continuous sample paths. This is a consequence of [2] and [6, Remark 2.3]. The Markov process can be regarded as a B-valued Brownian motion and in finite dimensions reduces to the usual Brownian motion.

The *Green measure* on B is defined on Borel sets A by

$$G(A) = \int_0^\infty p_t(A)\, dt.$$

For a bounded Borel function f: $B \to R$, the *potential of* f is the function $Gf(x) = \int_B f(x + y)\, G(dy)$. It has been shown that if f has bounded support, then Gf is bounded. If f is also Lip 1, then Gf is the solution of a generalized Poisson equation $\frac{1}{2}\Delta u = -f$.

2. **Definitions and statement of results.** Henceforth H and B will denote the first and second terms of an abstract Wiener space; $|\cdot|$ and $\|\cdot\|$ will be their respective norms. U will be an open set in B.

If V, W are Banach spaces and $\mathcal{L}(V, W)$ is the space of bounded linear operators $V \to W$, a function f: $V \to W$ is Fréchet differentiable at x if there exists $L \in \mathcal{L}(V, W)$ such that

$$\|f(x + y) - f(x) - L(y)\|_W = o(\|y\|_V).$$

In the case where V is the space B, we denote the derivative L by $D_B f(x)$ and call f *B-differentiable.*

On an abstract Wiener space, there is a weaker notion of differentiation which has proved to be important. A function f: $B \to W$ is *H-differentiable* at x (or differentiable in H-directions) if there exists $L \in \mathcal{L}(H, W)$ such that $\|f(x + h) - f(x) - L(h)\|_W = o(|h|)$. We denote L by $Df(x)$. It is possible for a function to be differentiable in H-directions but not even continuous in the norm topology on B. Of course, any such function will be *H-continuous*; that is, $\|f(x + h) - f(x)\|_W \to 0$ as $|h| \to 0$. There are Banach spaces where there are no B-differentiable $\mathbf{R}$-valued functions with bounded support [1], [13], but we will show that for any open set U contained in B there are H-differentiable functions with support contained in U.

Define inductively the Banach spaces $W_0 = \mathbf{R}$, $W_k = \mathcal{L}(H, W_{k-1})$ and let $|\cdot|_k$ be the usual operator norm on W_k. We say f: $B \to \mathbf{R}$ is H-differentiable n times on U if, for all $k \leq n$, $D^k f(x)$ is the Fréchet derivative of the function $D^{k-1}f$: $U \to W_{k-1}$, which maps $x \rightsquigarrow D^{k-1}f(x)$. If the functions $D^k f$ are continuous (where U has the relative B-topology and W_k the norm topology) for all

$k \leq n$, then f is *n times continuously H-differentiable* on U. It is easy to show that if $f\colon B \to W$ is H-differentiable on an open ball containing x and $x + h$, and Df is H-continuous,

$$\text{(2.1)} \qquad \frac{f(x+h) - f(x)}{|h|} = Df(x + ah)\left(\frac{h}{|h|}\right) \quad \text{for some } a \in (0,1).$$

If $A \subset U$, then A is a *properly bounded subset of U* if A is a bounded set and $d(A, U^c) = \inf_{x \in A; y \in U^c} \|x - y\| > 0$. (If $U = B$, the second condition is dropped.)

(2.2) Definition. Let $D^{n(H)}(U) = \{f\colon B \to \mathbf{R} \mid f$ is H-differentiable n times on U and $\sup_{x \in U} |D^k f(x)|_k < \infty$, all $k \leq n\}$. Let $D_0^{n(H)}(U)$ be all function in $D^{n(H)}(U)$ whose supports are properly bounded in U. The set of *test functions on* U is $\mathfrak{D}_0(U) = \{f \in C_0^{\infty(H)}(U) \mid D^k f\colon B \longrightarrow W_k$ is a Lip 1 function, all $k\}$. We will denote by $\mathfrak{D}(U)$ those functions which are as smooth as test functions on U but which do not have properly bounded support in U.

If f is a test function on U and $h_1, \ldots, h_k$ are fixed elements of H, $\phi(x) = D^k f(x)(h_1, \ldots, h_k)$ is a test function on U: it is a consequence of the definition of Fréchet derivative and the W_k-spaces that $\phi(x)$ is $D^{\infty(H)}$ with support $\subset \operatorname{Supp} f$. Furthermore

$$\begin{aligned}
&|D^n\phi(x) - D^n\phi(y)|_n \\
&\quad = \sup_{|p_i|=1} |D^{n+k} f(x)(h_1, \ldots, h_k, p_1, \ldots, p_n) \\
&\qquad\qquad - D^{n+k} f(y)(h_1, \ldots, h_k, p_1, \ldots, p_n)| \\
&\quad \leq |h_1| \cdots |h_k| \, |D^{n+k} f(x) - D^{n+k} f(y)|_{n+k}
\end{aligned}$$

so the map $D^n\phi\colon B \to W_n$ is Lip 1. In this sense, the set of test functions is closed under differentiation.

(2.3) Definition. A sequence of test functions $\{f_n\}$ will be said to be $\mathfrak{D}_0$ *-Cauchy* in U, written $f_n \twoheadrightarrow$, if

(i) there exists a fixed properly bounded $K \subset U$ such that for all n, $\operatorname{Supp} f_n \subset K$,

(ii) for all k, $|D^k f_n(x) - D^k f_m(x)|_k \to 0$ pointwise and uniformly boundedly as $n, m \to \infty$.

The set $\mathfrak{D}_0(U)$ is not complete with respect to this convergence. In the next section we show that if $f_n \twoheadrightarrow f$ then f is infinitely H-differentiable; however neither f nor its derivatives are necessarily continuous in the B topology.

(2.4) Definition. The *distributions on* U are $\mathbf{R}$-valued linear functions T on $\mathfrak{D}_0(U)$. such that if $\{f_n\}$ is a $\mathfrak{D}_0$-Cauchy sequence of test functions on U, then $\{T(f_n)\}$ converges.

For the Markov process described in the introduction, define the random variable τ_r to be the exit time from $B_r(0)$; that is, $\tau_r(\omega) = \inf_{t>0}\{t\colon \|X_t(\omega)\| \geq r\}$.

Let $\pi_{r,0}(dy) = P_0\{X_{\tau_r} \in dy\}$ be the associated hitting measure. $\pi_{r,0}$ is a probability measure concentrated on $\{\|x\| = r\}$ and $\pi_{r,0}(dy) = \pi_{r,0}(-dy)$.

It is a consequence of Proposition 3.5 that if T is a distribution on U, the convolution $T * \pi_{r,0}$ is a distribution on $U_r = \{x \in U: d(x, U^c) > r\}$.

(2.5) Definition. A *distribution* T is *harmonic on* U if for all $f \in \mathfrak{D}_0(U)$,

$$\lim_{r \to 0} \frac{T * \pi_{r,0}(f) - T(f)}{r^2} = 0.$$

This expression is well-defined for sufficiently small r because Supp f is properly bounded in U.

The definition of a harmonic distribution is based on a generalization of the mean value property. There are other possible definitions; for instance, one might require that $T(\text{Trace } D^2 f) = 0$ for a certain class of functions f. The difficulty with such an approach is in finding a suitable class of functions which is closed under the operation $f \rightsquigarrow \text{Trace } D^2 f$ but is still large enough to be interesting. These approaches are equivalent in finite dimensions, but in infinite dimensions $D^2 f$ may exist but not be trace class.

(2.6) Definition. A *local measure* on an open set U in B is a set function ν defined on properly bounded Borel subsets of U, such that for properly bounded open W in U, ν is a finite Borel measure on W.

The main result of this paper is the following:

(2.7) Theorem. *If T is a harmonic distribution on U, then there exists a unique local measure ν on U such that for $f \in \mathfrak{D}_0(U)$, $T(f) = \int f(x)\, \nu(dx)$.*

3. **Test functions and distributions.** In this section we present further properties of test functions and distributions which relate directly to the proof of Theorem 2.7.

There are many test functions on U: it is known that if f is a real bounded Borel measurable function on B then $p_t f$ is infinitely H-differentiable and

$$D^k p_t f(x)(h_1, \ldots, h_k) = \int f(x+y) J_k(y, h_1, \ldots, h_k) p_t(dy)$$

with [4]

$$J_1(y,h) = (1/t) h(y),$$

$$J_n(y, h_1, \ldots, h_n) = (1/t) h_n(y) J_{n-1}(y, h_1, \ldots, h_{n-1}) - DJ_{n-1}(y, h_1, \ldots, h_{n-1})(h_n).$$

Here $h(y)$ is the function defined as the $L^2(p_t)$-limit of any sequence $\{h_n\}$ of elements of B^* which converge to h in H norm, and therefore in $L^2(p_t)$ norm; we define $Dh(y) = h$ and $D^2 h(y) = 0$. These definitions agree with the usual definitions of $Dh(y)$, $D^2 h(y)$ when $h \in B^*$ and otherwise are merely a formal notation for writing the recursion formula. We have the following estimate on the $L^2(p_t)$ norm of $J_k(y, \bar{h}) = J_k(y, h_1, \ldots, h_k)$.

(3.1) Lemma. $\sup_{|h_i|=1} \|J_k(y,\bar{h})\|_{L^2(p_t)} \leq M_k t^{-k}$, *where* M_k *is independent of* t.

Proof. $J_k(y,\bar{h})$ is a finite sum of terms of the form

$$\pm \frac{h_{i_1}(y)}{t} \cdots \frac{h_{i_m}(y)}{t} \frac{(h_{i_{m+1}}, h_{i_{m+2}})}{t} \cdots \frac{(h_{i_{k-1}}, h_{i_k})}{t}.$$

Repeated use of the Hölder inequality yields that the $L^2(p_t)$ norm of such a term is bounded by $Ct^{-m/2}t^{-(k-m)/2}|h_1|\cdots|h_k|$, with C independent of t. In this we have used the estimate $\|h\|_{L^{2j}(p_t)} \leq C_j\sqrt{t}|h|$, which follows from the fact that $h(\cdot)$ is a Gaussian random variable on (B,p_t), with mean 0 and variance $t|h|^2$.

(3.2) Proposition. *For any open set* U, *the set* $\mathfrak{D}_0(U)$ *contains nonzero functions.*

Proof. Pick f a bounded real-valued Lip 1 function on B with properly bounded support in U such that $\sup_x|f(x)| \geq 1$. Then for all k, $D^k p_t f$: $B \to W_k$ is a Lip 1 function:

$$\begin{aligned} |D^k p_t f(x) - D^k p_t f(y)|_k &\leq \sup_{|h_i|=1} \int |f(x+z) - f(y+z)|\,|J_k(z,\bar{h})|\,p_t(dz) \\ &\leq c\|x-y\|M_k t^{-k}. \end{aligned}$$

The function $p_t f$ does not necessarily have bounded support but $p_t f \to f$ uniformly as $t \to 0$ [6, Proposition 6]. Hence for t sufficiently small, $p_t f < \frac{1}{4}$ on $(\operatorname{Supp} f)^C$ and, for some x, $|p_t f(x)| > \frac{3}{4}$. Choose a function g: $\mathbf{R} \to \mathbf{R}$ such that $g \in C_0^\infty(R)$ and $g = 0$ on $[-\frac{1}{4}, \frac{1}{4}]$, and $g = 1$ on $[-1,-½] \cup [½, 1]$. Let $\phi(x) = g(p_t f(x))$, for some fixed t, sufficiently small. Then ϕ is a nonzero function with $\operatorname{Supp}\phi \subset \operatorname{Supp} f$ and $\phi = 1$ on $\{x: f(x) = 1\}$. By the chain rule $\phi(x)$ is infinitely H-differentiable with $D^k\phi(x)(h_1,\ldots,h_k)$ a finite sum of terms of the form

$$D^{n_1}g(p_t f(x))D^{n_2}p_t f(x)(h_{i_1},\ldots,h_{i_{n2}})\cdots D^{n_j}p_t f(x)(h_{i_j},\cdots,h_k).$$

Since the composition and product of bounded Lip 1 functions are bounded and Lip 1, $D^k\phi$ is bounded and Lip 1. Hence ϕ is a nontrivial test function.

(3.3) Remark. Let μ be a finite Borel measure on B and let $f \in D^{n(H)}(B)$ with $D^k f$: $B \to W_k$ a measurable and H-continuous map, for all $k \leq n$. If we define $\mu * D^k f(x)$ by $\mu * D^k f(x)(\bar{h}) = \int D^k f(x-y)(\bar{h})\mu(dy)$ then $\mu * D^k f(x) \in W_k$, all $k \leq n$, and

$$\begin{aligned} |\mu * D^k f(x)|_k &\leq \int |D^k f(x-y)|_k |\mu|(dy) \\ &\leq \sup_x |D^k f(x)|_k |\mu|(B). \end{aligned} \tag{3.4}$$

Furthermore $D^k(f * \mu) = \mu * D^k f$. This is true for $k = 0$ and if it is true for $j < k$ then

$$\frac{1}{|h|}|D^{k-1}(f*\mu)(x+h)-D^{k-1}(f*\mu)(x)-\mu*D^kf(x)(h)|_{k-1}$$

$$\le \frac{1}{|h|}\sup_{|p_i|=1}\int |D^{k-1}f(x+h-y)(\bar{p})-D^{k-1}f(x-y)(\bar{p})$$

$$-D^kf(x-y)(\bar{p},h)|\,|\mu|(dy)$$

$$=\sup_{|p_i|=1}\int \left|D^kf(x-y+\alpha h)\left(\bar{p},\frac{h}{|h|}\right)-D^kf(x-y)\left(\bar{p},\frac{h}{|h|}\right)\right|\,|\mu|(dy)$$

where α depends on $\bar{p}$

$$\le \int \sup_{\alpha\in(0,1)}|D^kf(x-y+\alpha h)-D^kf(x-y)|_k|\mu|(dy)$$

$\to 0$ as $|h|\to 0$ by dominated convergence and H-continuity of D^kf.

In particular, if f is a test function on $U_r=\{x\in U\colon d(x,U^c)>r\}$ and μ is a finite Borel measure concentrated on $\overline{B_r(0)}$, then $\mu*f\in \mathfrak{D}_0(U)$: one needs only check that the map $x \rightsquigarrow D^k(\mu*f)(x)$ is Lip 1 from B to W_k. But (3.4) gives

$$|D^k(f*\mu)(x)-D^k(f*\mu)(y)|_k$$

$$\le \int |D^kf(x-z)-D^kf(y-z)|_k|\mu|(dz)\le c\|x-y\|\,|\mu|(B).$$

(3.5) Proposition. *If μ is a bounded Borel measure concentrated on $\{\|x\|\le r\}$ and T is a distribution on U, then $T*\mu$ is a distribution on U_r, where we define $T*\mu(f)=T(\mu*f)$.*

Proof. We have just shown that if $f\in\mathfrak{D}_0(U_r)$, $\mu*f\in\mathfrak{D}_0(U)$; hence $T*\mu$ is well-defined. It remains to show that if $\{f_n\}\twoheadrightarrow$ on U_r then $\{\mu*f_n\}\twoheadrightarrow$ on U. But this follows from (3.4) and dominated convergence.

(3.6) Proposition. *If $f\in\mathfrak{D}_0(B)$ and h is a function on* **R** *integrable with respect to Lebesgue measure, then $F(x)=\int_0^\infty h(t)p_tf(x)\,dt\in\mathfrak{D}(B)$.*

Proof. This follows from Remark 3.3 and the observation that $F(x)=I*f(x)$, where I is the finite measure defined on Borel sets by $I(A)=\int_0^\infty h(t)p_t(A)\,dt$.

(3.7) Proposition. *Let A be a subset of an open set U. If $f,g\in\mathfrak{D}(A)$, then $F(x)=(Df(x),Dg(x))\in\mathfrak{D}(A)$. If* Supp $Df\subset A$ *and is properly bounded in U, then $F\in\mathfrak{D}_0(U)$.*

Proof. Clearly F is bounded and has support contained in Supp Df. It is Lip 1 because

$$|F(x)-F(y)|\le |Df(x)|_1|Dg(x)-Dg(y)|_1$$

$$+|Df(x)-Df(y)|_1|Dg(y)|_1$$

and both Df, Dg are bounded and Lip 1 on A.

Next we show that

$$DF(x)(h) = (D^2f(x)(h), Dg(x)) + (Df(x), D^2g(x)(h)).$$

The right-hand side is linear in h since we are only concerned with real Hilbert spaces; furthermore

$$\begin{aligned} &|(D^2f(x), Dg(x)) + (Df(x), D^2g(x))|_1 \\ &\qquad \leq |D^2f(x)|_2|Dg(x)|_1 + |Df(x)|_1|D^2g(x)|_2. \end{aligned} \tag{3.8}$$

We have

$$\begin{aligned} &(1/|h|)|F(x+h) - F(x) - (D^2f(x)(h), Dg(x)) + (Df(x), D^2g(x)(h))| \\ &\qquad \leq (1/|h|)|(Df(x+h), Dg(x+h) - Dg(x)) - (Df(x), D^2g(x)(h))| \\ &\qquad\quad + (1/|h|)|(Df(x+h) - Df(x) - D^2f(x)(h), Dg(x))|. \end{aligned}$$

Since $(1/|h|)|Df(x+h) - Df(x) - D^2f(x)(h)|_1 \to 0$ as $|h| \to 0$ by hypothesis, the second term $\to 0$ as $|h| \to 0$.

The first term is

$$\begin{aligned} &\leq (1/|h|)|(Df(x+h) - Df(x), Dg(x+h) - Dg(x))| \\ &\quad + (1/|h|)|(Df(x), Dg(x+h) - Dg(x) - D^2g(x)(h))|. \end{aligned}$$

Again the second part $\to 0$ as $|h| \to 0$ by the existence of $D^2g(x)$. Since Df, Dg are Lip 1, and the H norm dominates the B norm, the first part is dominated by $(1/|h|)k\|h\|^2 \leq k'|h|^2/|h| \to 0$ as $|h| \to 0$. Thus F is H-differentiable on A and it follows from (3.8) that $\sup_x|Df(x)|_1 < \infty$. It is quite straightforward to prove that $x \rightsquigarrow (D^2f(x), Dg(x))$, hence $x \rightsquigarrow DF(x)$, is Lip 1: $B \to W_1$, using the smoothness properties of D^if, D^ig, $i = 1, 2$.

Similar arguments show that

$$D^kF(x)(\bar{h}) = \sum_{\text{finite}} (D^if(x)(h_{j1}, \ldots, h_{ji-1}), D^{k+2-i}g(x)(h_{ji}, \ldots, h_{jk}))$$

and $x \rightsquigarrow D^kF(x)$ is bounded and Lip 1.

(3.9) Proposition. *If a sequence of test functions $\{f_n\}$ is $\mathfrak{D}_0$-Cauchy on U, then the limit function $f \in D_0^{\infty(H)}(U)$.*

Proof. For each k there is a collection of elements $A(x) \in W_k$ with $D^kf_n(x) \to_n A(x)$ in W_k norm pointwise and boundedly. We will show that $D^kf(x) = A(x)$. It is true for $k = 0$ and if it is true for $k - 1$ then it is sufficient to prove that

$$\sup|D^{k-1}f(x+h)(\bar{p}) - D^{k-1}f(x)(\bar{p}) - A(x)(\bar{p}, h)| = o(|h|).$$

Here and below the supremum is taken over a countable dense subset of

$\{\bar{p} \in H^{k-1}: |p_i| = 1\}$ to ensure measurability. If we let $\phi_n(s) = D^{k-1}f_n(x + sh)(\bar{p})$, then $\phi_n'(s) = D^k f_n(x + sh)(\bar{p}, h)$ and $\phi_n''(s) = D^{k+1}f_n(x + sh)(\bar{p}, h, h)$. If $M = \sup_n \sup_x |D^{k+1}f_n(x)|_{k+1}$, then the mean value theorem gives $|\phi_n'(s) - \phi_n'(0)| \le |s| M |h|^2$. But $\phi_n'(s) \to_n A(x + sh)(\bar{p}, h)$ pointwise and boundedly. Since M is independent of n and also of $\bar{p}$, we have

$$\sup_{\bar{p}} |A(x + sh)(\bar{p}, h) - A(x)(\bar{p}, h)| \le M|s||h|^2.$$

Using

$$\begin{aligned} &D^{k-1}f(x + h)(\bar{p}) - D^{k-1}f(x)(\bar{p}) \\ &\qquad = \lim_n \phi_n(1) - \phi_n(0) = \lim_n \int_0^1 \phi_n'(s)\,ds = \int_0^1 A(x + sh)(\bar{p}, h)\,ds \end{aligned}$$

we get

$$\begin{aligned} \sup |D^{k-1}f(x + h)(\bar{p}) &- D^{k-1}f(x)(\bar{p}) - A(x)(\bar{p}, h)| \\ &\le \int_0^1 \sup |A(x + sh)(\bar{p}, h) - A(x)(\bar{p}, h)|\,ds \\ &\le \int_0^1 Ms|h|^2 ds = \tfrac{1}{2}M|h|^2 = o(|h|). \end{aligned}$$

It is clear that $\operatorname{Supp} f$ is properly bounded in U and the norms of the derivatives are uniformly bounded.

We conclude this section with an example of a distribution harmonic on $B - \{0\}$, namely, the Green measure $G(dx) = \int_0^\infty p_t(dx)\,dt$. For $f \in \mathfrak{D}_0(B)$, $G(f) = \int f(x)\,G(dx)$ is well defined since $\operatorname{Supp} f$ is bounded. It is easy to see that if $f_n \longrightarrow\!\!\!\!\rightarrow$ on B, $G(f_n)$ converges; hence G is a distribution on B.

To show that G is harmonic on $B - \{0\}$, we will show that for A a properly bounded Borel subset of $B - \{0\}$ and $r < d(0, A)$, $G * \pi_{r,0}(A) = G(A)$. This is sufficient because G is a measure and a test function on $B - \{0\}$ can be approximated pointwise and boundedly by simple functions with uniformly bounded support.

$$\begin{aligned} G * \pi_{r,0}(A) &= \int_B G(A - x)\pi_{r,0}(dx) \\ &= \int_B \int_0^\infty P_x(X_t \in A)\,dt\,\pi_{r,0}(dx) \\ &= \int_\Omega \int_0^\infty P_{X_{\tau_r}}(X_t \in A)\,dt\,P_0 \\ &= \int_0^\infty \int_\Omega P_{X_{\tau_r}}(X_t \in A)\,P_0\,dt. \end{aligned}$$

We can interchange the integrals because $g(t, \omega) = P_{X_{\tau_r(\omega)}}(X_t \in A)$ is jointly measurable in t and ω for fixed Borel A: the map $(\omega, t) \rightsquigarrow (X_{\tau_r(\omega)}, t)$ is measurable, and when A is a cylinder set it is clear from the formula for $p_t(A - x)$ that

$P_x(X_t \in A)$ is jointly measurable in x and t. But $\{A \in \mathcal{B}: P_x(X_t \in A)$ is jointly measurable in $t, x\}$ is a monotone class containing the field of cylinder sets and is, therefore, the Borel field.

Using the strong Markov property,

$$G * \pi_{r,0}(A) = \int_0^\infty \int_\Omega E(\chi_{\{X_t \circ \theta_{\tau_r} \in A\}} | \sigma_{\tau_r}) P_0 dt$$
$$= \int_0^\infty \int_\Omega \chi_A(X_t \circ \theta_{\tau_r}) P_0 dt.$$

We again interchange integrals, using the fact that $X_{t+\tau_r(\omega)}(\omega)$ is continuous in t for fixed ω and measurable in ω for fixed t, hence is jointly measurable in t and ω.

$$= \int_\Omega \int_0^\infty \chi_A(X_{t+\tau_r})\, dt\, P_0,$$

Letting $s = t + \tau_r(\omega)$,

$$= \int_\Omega \int_{\tau_r(\omega)}^\infty \chi_A(X_s)\, ds\, P_0(\omega)$$
$$= \int_\Omega \int_0^\infty \chi_A(X_s)\, ds\, P_0.$$

Since for $s < \tau_r$, $\chi_s \notin A$ because $A \subset B_r(0)^C$ and P_0 is concentrated on paths starting at 0. Thus we get

$$G * \pi_{r,0}(A) = \int_0^\infty P_0\{X_s \in A\}\, ds = G(A).$$

4. Approximation of distributions by measures. In the finite dimensional theory of distributions, the convolution of a test function with a distribution can be defined and shown to be a C^∞ function. This observation may be applied to a sequence of test functions which converge to $\delta_{\{0\}}$ appropriately to show that a distribution can be approximated in a distribution sense by a sequence of functions. The purpose of this section is to construct a sequence of measures which approximate $\delta_{\{0\}}$ and which yield a local measure when convolved with a distribution on an abstract Wiener space.

(4.1) Lemma. *Let z be a positive test function on $B_r(0)$ with $z(y) = z(-y)$ and $z(0) = 1$. Let $\mu_t(dy) = z(y)p_t(dy)$.*

(i) *If f is a* Lip 1 *function with bounded support $\subset U_r$, $\mu_t * f \in \mathfrak{D}_0(U)$.*

(ii) *If $\{f_n\}$ is a sequence of* Lip 1 *functions with $\bigcup_n$* Supp (f_n) *a bounded subset of U_r and $f_n \to f$ pointwise and boundedly, $\mu_t * f \longrightarrow\!\!\!\!\rightarrow$ as $n \longrightarrow \infty$.*

(iii) *If $f \in \mathfrak{D}_0(U)$ with support in U_r, $\mu_t * f \longrightarrow\!\!\!\!\rightarrow f$ as $t \longrightarrow 0$.*

Proof. Using the notation of §3, we first show

$$D\mu_t * f(x)(h) = \int f(x+y)[z(y)J_1(y,h) - Dz(y)(h)]p_t(dy).$$

Noting that $\mu_t * f(x+h) = \int f(x+u)z(u-h)p_t(h, du)$, we have

$$\begin{aligned}\mu_t * f(x+h) - \mu_t * f(x) &= \int f(x+y)[z(y-h) - z(y)][p_t(h,dy) - p_t(dy)] \\ &\quad + \int f(x+y)z(y)[p_t(h,dy) - p_t(dy)] \\ &\quad + \int f(x+y)[z(y-h) - z(y)]p_t(dy).\end{aligned}$$

Then

$$\begin{aligned}&\left|\mu_t * f(x+h) - \mu_t * f(x) - \int f(x+y)[z(y)J_1(y,h) - Dz(y)(h)]p_t(dy)\right| \\ &\qquad \le \left|\int f(x+y)[z(y-h) - z(y)][p_t(h,dy) - p_t(dy)]\right| \\ &\qquad\quad + \left|\int f(x+y)[z(y-h) - z(y) + Dz(y)(h)]p_t(dy)\right| \\ &\qquad\quad + \left|\int f(x+y)z(y)[p_t(h,dy) - p_t(dy) - J_1(h,y)]p_t(dy)\right|.\end{aligned}$$

For each x, let $g_x(y) = f(y)z(y-x)$, a bounded measurable function. Then $p_t g_x$ is H-differentiable everywhere, which shows that the third term $= o(|h|)$. Let I_1, I_2 be the first and second integrals, respectively.

$$\frac{|I_2|}{|h|} \le \|f\|_\infty \int |z(y-h) - z(y) - Dz(y)(-h)|\, p_t(dy).$$

By (2.1), for $|h| \le 1$ the integrand is bounded by $2\sup_x |Dz(x)|_1 < \infty$. Since z is H-differentiable, we use dominated convergence to show $|I_2| = o(|h|)$. From [6, Proposition 9],

$$\begin{aligned}|p_t(h,dy) - p_t(dy)|(B) &\le \int |\exp((-|h|^2 + 2h(y))/2t) - 1| \\ &\le (\exp(-|h|^2/t) - 1)^{1/2} \to 0 \quad \text{as } |h| \to 0.\end{aligned}$$

Thus

$$\frac{|I_1|}{h} \le \|f\|_\infty \sup_x |Dz(x)|_1\, |p_t(h,dy) - p_t(dy)|(B) \to 0 \quad \text{as } |h| \to 0.$$

Similar calculations show

$$D^k \mu_t * f(x)(\bar{h}) = \int f(x+y) P_k(y, \bar{h})\, p_t(dy)$$

where $P_k(y, \bar{h})$ is a finite sum of terms of the form

$$\pm D^n z(y)(h_{i1}, \ldots, h_{in}) J_{k-n}(y, h_{i_{n+1}}, \ldots, h_{ik}).$$

Hence $\mu_t * f \in D^{\infty(H)}(B)$, once we show that, for all k, $\sup_x |D^k \mu_t * f(x)|_k < \infty$.

$$
(4.2)\qquad
\begin{aligned}
|D^k\mu_t * f(x)|_k &\le \|f\|_\infty \sum_{\text{finite}} \int \sup_{|h_i|=1} |D^n z(y)(h_{i1},\ldots,h_{in})| \\
&\qquad\qquad \cdot |J_{k-n}(y,h_{i_{n+1}},\ldots,h_{ik})| p_t(dy) \\
&\le \|f\|_\infty \sum_{\text{finite}} \int |D^n z(y)|_n \sup_{|h_i|=1} |J_{k-n}(y,\bar{h})| p_t(dy) \\
&\le \|f\|_\infty \sum_{\text{finite}} \sup_y |D^n z(y)|_n M_{k-n} t^{-(k-n)} = K\|f\|_\infty .
\end{aligned}
$$

Since the bound is independent of x, we are done.

Clearly Supp $\mu_t * f \subset$ Supp μ_t + Supp f is a properly bounded subset of U, since μ_t is concentrated on a properly bounded subset of $B_r(0)$. One repeats the calculations above to show that

$$|D^k\mu_t * f(x) - D^k\mu_t * f(x_1)|_k \le c\|x - x_1\|_k,$$

where c is the Lip 1 constant of f. Thus each derivative $D^k\mu_t * f$ is Lip 1: $B \to W_k$ and we have shown $\mu_t * f \in \mathfrak{D}_0(U)$.

To prove (ii) we note that $\bigcup_n$ Supp $\mu_t * f_n$ is a properly bounded subset of U and we must show that for all k, $D^k\mu_t * f_n(x)$ converges in W_k norm pointwise and boundedly as $n \to \infty$.

$$
\begin{aligned}
&|D^k\mu_t * f_n(x) - D^k\mu_t * f_m(x)|_k \\
&\quad \le \sup_{|h_i|=1} \int |f_n(x+y) - f_m(x+y)|\,|P_k(y,\bar{h})| p_t(dy) \\
&\quad \le \left[\int |f_n(x+y) - f_m(x+y)|^2 p_t(dy)\right]^{1/2} \sup_{|h_i|=1} \left[\int |P_k(y,\bar{h})|^2 p_t(dy)\right]^{1/2}.
\end{aligned}
$$

The first factor $\to_{n,m} 0$ by Lebesgue dominated convergence, while estimates similar to (4.2) show that the second factor is bounded. The proof is concluded by using (4.2) to show

$$\sup_n \sup_x |D^k\mu_t * f_n(x)|_k \le K \sup_n \|f_n\|_\infty .$$

To prove (iii) observe that if $f \in \mathfrak{D}_0(U_r)$ then $\bigcup_{t>0}$ Supp $(\mu_t * f) \subset$ Supp f + $B_r(0)$, a properly bounded subset of U. It follows that $\mu_t * f \twoheadrightarrow f$ if $|D^k\mu_t * f(x) - D^k f(x)|_k \to 0$ pointwise and boundedly as $t \to 0$. By the regularity of f and Remark 3.3, $D^k\mu_t * f = \mu_t * D^k f$. Hence

$$
\begin{aligned}
|D^k\mu_t * f(x) - D^k f(x)|_k &= \sup_{|h_i|=1} \left|\int D^k f(x+y)(\bar{h}) z(y) - D^k f(x)(\bar{h}) p_t(dy)\right| \\
&\le \sup_{|h_i|=1} \int |D^k f(x+y)(\bar{h}) z(y) - D^k f(x+y)(\bar{h})| p_t(dy) \\
&\qquad + \int |D^k f(x+y) - D^k f(x)|_k p_t(dy) \\
&\le \sup_x |D^k f(x)|_k \int |z(y) - 1| p_t(dy) + \int |D^k f(x+y) - D^k f(x)|_k p_t(dy).
\end{aligned}
$$

In both integrals, the integrand is a continuous function of y and $= 0$ at $y = 0$. As $t \to 0$, $p_t\{\|y\| \geq \delta\} \to 0$ for all $\delta > 0$; hence both integrals $\to 0$ as $t \to 0$. Furthermore, the convergence is bounded by a constant $\cdot \sup_x |D^k f(x)|_k$.

(4.3) Theorem. *Every distribution can be approximated by local measures in the following sense: for each $r > 0$, there is a family $\{\nu_t\}$ of set functions defined on bounded Borel subsets of U_r which are measures when restricted to bounded open subsets of U_r. If T is a distribution on U and f is a test function with* Supp $f \subset U_r$, *then $\int f(x)\, \nu_t(dx) \to T(f)$ as $t \to 0$.*

Before proving this theorem we need the following lemma, which is an application of the Daniell integral.

(4.4) Lemma. *Let W be an open bounded subset of B and $L(W) = \{f\colon B \to R \mid f$ is* Lip 1 *on W and bounded$\}$. If S is a real linear functional on $L(W)$ satisfying*

(4.5) *for every sequence $\{f_n\}$ in $L(W)$ which converges pointwise and boundedly, $\{S(f_n)\}$ is Cauchy*

then there exists a unique outer regular finite Borel measure ν on $(W, \mathcal{B} \cap W)$ such that $S(f) = \int_W f\, d\nu$.

Proof. This is an application of [11, Proposition II-7-1], provided we show that $L(W)$ is a Riesz space and S is the difference of two positive linear functionals S^+, S^- such that

(4.6) if $f_n \downarrow 0$ pointwise and boundedly, $S^{\pm}(f_n) \downarrow 0$.

In particular, ν is a measure on the σ-field generated by sets A whose characteristic function is the limit of a monotone increasing sequence of $L(W)$ functions. But any open set $A \subset W$ has this property: let $f_n(x) = 1 \wedge nd(x, A^c)$. As shown below, f_n is Lip 1 on W; clearly $f_n = 0$ on A^c and if $x \in A$, $f_n(x) \uparrow 1$. Hence $\mathcal{B} \cap W \subset \{\nu\text{-measurable sets}\}$.

Since $L(W)$ is a linear space, it is a Riesz space if it is closed under the operations $\vee$, $\wedge$. Let $f, g \in L(W)$. To calculate $f \vee g(x) - f \vee g(y)$ one needs to consider two cases, interchanging f and g as necessary:

(i) $f \vee g(x) = f(x)$ and $f \vee g(y) = f(y)$,

(ii) $f \vee g(x) = f(x)$ and $f \vee g(y) = g(y)$.

In the first case we use the fact that f is Lip 1 on W to get the necessary inequality. In the second case we have $f(x) \geq g(x)$ and $g(y) \geq f(y)$, and we observe that

$$\begin{aligned} |f(x) - g(y)| &= f(x) - g(y) \leq f(x) - f(y) = |f(x) - f(y)| \qquad \text{if } f(x) \geq g(y) \\ &= g(y) - f(x) \leq g(y) - g(x) = |g(y) - g(x)| \qquad \text{if } f(x) < g(y). \end{aligned}$$

In both cases, $|f \vee g(x) - f \vee g(y)| \leq c\|x - y\|$, where c is the larger of the Lip 1 constants for f, g.

For closure under $\wedge$, observe that $f \wedge g = -(-f \vee -g)$.

It remains to show that $S = S^+ - S^-$, where $S^\pm$ are positive linear functionals satisfying (4.6). If $0 \leq f \in L(W)$, we define $S^+(f) = \sup_{0 \leq h \leq f; h \in L(W)} S(h) \geq S(0) = 0$. $S^+(f) < \infty$: Let $C(W)$ be the Banach space of bounded continuous functions on W with the sup norm $\| \; \|_\infty$. $L(W) \subset C(W)$ as a subspace and S is a linear functional on $L(W)$ continuous with respect to $\| \; \|_\infty$ since uniform convergence implies pointwise, bounded convergence. Extending S to a continuous linear functional on $C(W)$, there exists K, depending only on S, such that $|S(f)| \leq K\|f\|_\infty$. Hence $S^+(f) \leq K\|f\|_\infty < \infty$.

For $0 \leq f, g \in L(W)$ and $c \geq 0$, a standard arqument shows $S^+(cf + g) = cS^+(f) + S^+(g)$. Extend S^+ to $L(W)$ by $S^+(f) = S^+(f \vee 0) - S^+(-f \vee 0)$. Again, it is straightforward to show S^+ is a positive linear functional on $L(W)$. If $0 \leq f_n \uparrow f$ and $f \in L(W)$, then by positivity $S^+(f_n)\uparrow$. Suppose $\lim_n S^+(f_n) = a < S^+(f)$. Then there exists $0 \leq h \leq f$, $h \in L(W)$, such that $a < S(h) \leq S^+(f)$. But $h \wedge f_n \leq f_n$ and $h \wedge f_n \to h$ as $n \to \infty$ pointwise and boundedly. Therefore $S(h \wedge f_n) \to S(h)$ by hypothesis. But $S(h \wedge f_n) \leq S^+(f_n) \leq a$, which is a contradiction. From this it follows that S satisfies (4.6), since if $f_n \downarrow 0, f_1 - f_n \uparrow f_1$ and $S^+(f_1 - f_n) = S^+(f_1) - S^+(f_n) \uparrow S^+(f_1)$.

Define $S^-(f) = S^+(f) - S(f)$. S^- is positive since for $f \geq 0$, $S^+(f) \geq S(f)$; S^- is a linear functional on $L(W)$; and S^- clearly has the necessary continuity conditions since S^+, S do.

Proof of Theorem. Let W be a bounded open subset of U_r. Let z and μ_t be as described in Lemma 4.1. We will show that $T * \mu_t$ is a measure on W. Let $q_k\colon \mathbf{R} \to \mathbf{R}$ be a continuous piecewise linear function bounded above by 1, $\equiv 0$ on $x < 1/k$ and $\equiv 1$ on $x > 2/k$. Then for any $\phi \in L(W)$ we can define a sequence of functions by $h_k(x) = \phi(x)q_k(d(x, W^c))$. The h_k are bounded Lip 1 functions whose supports are contained in W and they converge pointwise and boundedly on W to ϕ; in fact for each $x \in W$, for $k > 2/d(x, W^c)$, $h_k(x) = \phi(x)$.

Define a linear function S on $L(W)$ by $S(\phi) = \lim_k T(\mu_t * h_k)$. This is well defined by Lemma 4.1 and is independent of the q_k: if h'_k is another sequence of bounded Lip 1 functions with support contained in W and if $h'_k \to \phi$ pointwise and boundedly on W, then $\mu_t * h_k - \mu_t * h'_k \twoheadrightarrow 0$.

We will show that S satisfies (4.5). Assume $\phi_n \in L(W)$ converge pointwise and boundedly to ϕ. (Note that ϕ may not be in $L(W)$.) For each ϕ_n, let $\{h_{n,k}\}$ be the corresponding sequence of $L(W)$ functions with bounded support described above. If $k(n) \geq n$, all n, then $h_{n,k(n)} \to \phi$ pointwise and boundedly on W: for $x \in W$ and $k(n) \geq n > N(x) > 2/d(x, W^c)$, $h_{n,k(n)}(x) = \phi_n(x)$. The uniform boundedness follows from $h_{n,k} \leq \phi_n$, all k, n. Hence $a = \lim_n T(\mu_t * h_{n,k(n)})$ is independent of the choice of $\{k(n)\}$, provided $k(n) \geq n$, all n. Given $\varepsilon > 0$, there is for each n a number $m(n)$ such that for $k \geq m(n)$, $|S(\phi_n) - T(\mu_t * h_{n,k})| < \varepsilon$. Without loss of generality assume $m(n) \geq n$. For the sequence $\{h_{n,m(n)}\}$, there exists N such that for $n > N$, $|a - T(\mu_t * h_{n,m(n)})| < \varepsilon$. Thus $|a - S(\phi_n)| < 2\varepsilon$ for $n > N$, and $\{S(\phi_n)\}$ is Cauchy.

By Lemma 4.4, there exists a unique outer regular finite Borel measure ν_t on W such that $S(\phi) = \int_W \phi(y)\nu_t(dy)$. In particular, if $f \in \mathfrak{D}_0(U)$ with support contained in W, $\int f(y)\nu_t(dy) = S(f) = T(f * \mu_t) = T * \mu_t(f)$. It is clear from the definition of S that if V is another bounded open subset of U_r with corresponding measure ν_t', then $\nu_t|_{W \cap V} = \nu_t'|_{W \cap V}$.

Finally, from Lemma 4.1(iii), as $t \to 0$, $T * \mu_t(f) \to T(f)$ for $f \in \mathfrak{D}_0(U)$ with support in U_r.

5. Representation theorems for harmonic measures and distributions.

(5.1) Definition. A local measure is harmonic on U if for f a bounded Lip 1 function with properly bounded support in U

$$\lim_{r\to 0}\frac{1}{r^2}\left[\int f * \pi_{r,0}(x)\,\nu(dx) - \int f(x)\,\nu(dx)\right] = 0.$$

It is clear from the definitions that if ν is a harmonic local measure on U, then it is a harmonic distribution on U. Conversely, if T is a harmonic distribution on U and we choose μ_t as in Lemma 4.1, then $T * \mu_t$ is a harmonic local measure on U_r: if f is bounded, Lip 1 with properly bounded support in U_r then for δ sufficiently small, $f * \mu_t$ and $f * \pi_{\delta,0} * \mu_t \in \mathfrak{D}_0(U)$ and

$$\begin{aligned}\frac{1}{\delta^2}&\left[\int f * \pi_{\delta,0}(x)\,T * \mu_t(dx) - \int f(x)\,T * \mu_t(dx)\right]\\ &= \frac{1}{\delta^2}T(f * \pi_{\delta,0} * \mu_t - f * \mu_t)\\ &= \frac{1}{\delta^2}T * \pi_{\delta,0}(f * \mu_t) - T(f * \mu_t) \to 0 \quad \text{as } \delta \to 0.\end{aligned}$$

Following [6], let the *generalized Laplacian* of a function ϕ be

$$\Delta\phi(x) = \lim_{r\to 0}\frac{\phi * \pi_{r,0}(x) - \phi(x)}{cr^2}$$

whenever the limit exists. Here $c = \frac{1}{2}E_0(\tau_{r,0})$, with E_0 the expectation with respect to the probability measure P_0. Roughly speaking, if ν is a harmonic local measure and ϕ is a function such that $\Delta\phi(x)$ exists pointwise and boundedly, then $\int \Delta\phi\, d\nu = 0$. This is the key to the first representation theorem in this section, concerning harmonic local measures. First, however, we will consider conditions which ensure that $\Delta\phi$ exists pointwise and boundedly.

(5.2) Proposition. *Let ϕ be a real* Lip 1 *function of bounded support on B satisfying*

(i) $\phi \in D_0^{2(H)}(B)$,

(ii) $D^2\phi(x)$ *is a trace class operator, all x,*

(iii) *the map $B \to$ {trace class operators} given by $x \rightsquigarrow D^2\phi(x)$ is bounded and continuous.*

Then $(1/cr^2)[\phi * \pi_{r,0}(x) - \phi(x)] \to_{r\to 0} \operatorname{Trace} D^2\phi(x)$ *pointwise and boundedly.*

This proposition gives sufficient conditions for a function to be in the domain of the characteristic operator for the Brownian motion. In [12] it is shown that if (iii) is strengthened to uniform continuity then the function is in the domain of the infinitesimal generator of the process.

Proof. Let $f(x) = -\text{Trace } D^2\phi(x)$, a bounded continuous function with bounded support. Let $v(t,x) = 2p_t\phi(x)$. By [6, Theorem 3], $v(t, x)$ is jointly uniformly continuous on $[0, \infty) \times B$ and $dv(t,x)/dt = \text{Trace } D^2p_t\phi(x)$ for $t > 0$.

But Trace $D^2p_t\phi(x) = p_t * \text{Trace } D^2\phi(x)$: the map $x \rightsquigarrow D^2\phi(x)$ is continuous: $B \to W_2$ since the trace class norm dominates the operator norm, so we have $D^2p_t\phi(x) = p_t * D^2\phi(x)$ by Remark 3.3. Let $\|\cdot\|_{\mathrm{Tr}}$ be the trace class norm. Then

$$\begin{aligned} \|D^2p_t\phi(x)\|_{\mathrm{Tr}} &= \sup \sum_n |p_t D^2\phi(x)(f_n, g_n)| \\ &\leq \sup \int \sum_n |D^2\phi(x+y)(f_n, g_n)| p_t(dy), \end{aligned}$$

where both suprema are taken over all possible complete orthonormal systems $\{f_n\}, \{g_n\}$

$$\begin{aligned} &\leq \int \|D^2\phi(x+y)\|_{\mathrm{Tr}}\, p_t(dy) \\ &\leq \left\| \sup_x D^2\phi(x) \right\|_{\mathrm{Tr}}. \end{aligned}$$

Hence $D^2p_t\phi(x)$ is trace class; using the fact that if A is trace class and $\{e_i\}$ is a c.o.n.s. for H, Trace $A = \sum_i (Ae_1, e_i)$ one can show Trace $D^2p_t\phi(x) = p_t * \text{Trace } D^2\phi(x)$.

From (iv) we can show that Trace $D^2\phi(x)$ is a bounded continuous function of x. Therefore, for fixed x,

$$\frac{d}{dt}v(t,x) = \int \text{Trace } D^2\phi(x + \sqrt{t}y)p_1(dy)$$

is a continuous function of t on $(0, \infty)$. It follows that

$$2\phi(x) = \lim_{t\to 0} v(t,x) = \lim_{t\to\infty} v(t,x) - \int_0^\infty \frac{d}{dt}v(t,x)\,dt.$$

We have

$$\lim_{t\to\infty} v(t,x) = 2 \lim_{t\to\infty} \int \phi(x+y)p_t(dy) = 2 \lim_{t\to\infty} \int \phi(x + \sqrt{t}z)p_1(dz).$$

But this limit is 0 because the support of ϕ is bounded.

From this we get

$$\begin{aligned} 2\phi(x) &= -\int_0^\infty \text{Trace } D^2p_t\phi(x)\,dt = \int_0^\infty p_t(-\text{Trace } D^2\phi(x))\,dt \\ &= \int_0^\infty p_t f(x)\,dt = Gf(x). \end{aligned}$$

Since $|f(x)| \leq \sup_x \|D^2\phi(x)\|_{\mathrm{Tr}} \chi_{\{\mathrm{Supp}\,\phi\}}$ and f is continuous on B, we have from [6, Remark 3.6] that

$$\Delta\phi(x) = \tfrac{1}{2}\Delta Gf(x) = -f(x) = \text{Trace } D^2\phi(x)$$

and that, for all r,

$$\begin{aligned}\sup_x \left| \frac{1}{cr^2}(\phi * \pi_{r,0}(x) - \phi(x)) \right| \\ &\leq \|f\|_\infty + \frac{1}{E_0(\tau_{r,0})} E_0\left[\int_0^{\tau_{r,0}} |f(x+X_t) - f(x)|\,dt\right] \\ &\leq 3 \sup_x \|D^2\phi(x)\|_{\mathrm{Tr}}.\end{aligned}$$

Thus $(\phi * \pi_{r,0}(x) - \phi(x))/cr^2 \to \text{Trace } D^2\phi(x)$ pointwise and boundedly.

(5.3) Corollary. *If ϕ satisfies the hypothesis of Proposition* 5.2 *and ν is a local measure harmonic on* Supp ϕ *then* $\int \text{Trace } D^2\phi(x)\,\nu(dx) = 0$.

Not all test functions have trace class second derivatives. This is inconvenient. Nevertheless we can construct test functions which do satisfy the hypotheses of Proposition 5.2 and have the additional property that the trace of the second derivative is also a test function in a specified set.

(5.4) Lemma. *Given an open set W properly bounded in an open set U, there exists a function $g \in \mathfrak{D}_0(U)$ satisfying*

(i) $g \equiv 1$ *on W*,

(ii) $x \rightsquigarrow D^2 g(x)$: $B \to$ {*trace class operators*} *is bounded and continuous*,

(iii) Trace $D^2 g \in \mathfrak{D}_0(U)$.

Proof. The proof of Proposition 3.2 shows that there is a positive test function w on U with $w(x) = 1$ on W. Since $p_t w$ converges uniformly to w as $t \to 0$, choose T such that $\|p_t w - w\|_\infty < \frac{1}{4}$ for $t \leq T$. Choose $h \in C_0^\infty(R)$ such that $h(0) = 1$, $h(t) = 0$ if $|t| > T$. Let $\alpha = \int_0^\infty |h(t)|\,dt$. Then $I(x) = \int_0^\infty h(t) p_t w(x)\,dt < \frac{1}{4}\alpha$ for $x \notin U$ and $I(x) \in (\frac{3}{4}\alpha, \|w\|_\infty \alpha]$ for $x \in W$. Choose $\xi \in C_0^\infty(R)$ such that $\xi(x) = 0$ on $|x| \leq \frac{1}{4}\alpha$ and $\xi(x) = 1$ on $\frac{3}{4}\alpha \leq |x| \leqslant \|w\|_\infty \alpha$. Let $g(x) = \xi(I(x))$. Then $g = 1$ on W and it is a consequence of Proposition 3.6 that $g \in \mathfrak{D}_0(U)$.

We have

$$D^2 g(x)(h,k) = \xi''(I(x))DI(x)(h)DI(x)(k) + \xi'(I(x))D^2 I(x)(h,k).$$

By [4, Lemma 1.1], I a Lip 1 function $\Rightarrow DI(x) \in B^*$. Thus for arbitrary y, z in B, one can define

$$\langle A(x)\,y,\, z\rangle = DI(x)(y)DI(x)(z).$$

$A(x) \in \mathcal{L}(B, B^*)$ and its restriction to H is symmetric; therefore $A(x)$ is trace class [5]. From the polar decomposition of a symmetric trace class operator,

$$\begin{aligned}\|A(x)\|_{\mathrm{Tr}} &\le \sup_{\mathrm{cons}\{e_i\}} \sum_i |(A(x)e_i, e_i)| \\ &\le \sup_{\mathrm{cons}\{e_i\}} \sum_i |DI(x)(e_i)DI(x)(e_i)| \\ &\le |DI(x)|_1^2.\end{aligned}$$

Using the Schwartz inequality again,

$$\|A(x) - A(y)\|_{\mathrm{Tr}} \le |DI(x) + DI(y)|_1 |DI(x) - DI(y)|_1 \to 0 \quad \text{as } y \to x.$$

This establishes the boundedness and continuity of the first term of D^2g and for the second term we show, following [6, Theorem 3], that $D^2I(x)$ is trace class with $\|D^2I(x)\|_{\mathrm{Tr}} \le \|h\|_\infty \int_0^T \|D^2p_t w(x)\|_{\mathrm{Tr}}\,dt$. The integral is finite because $\|D^2p_t w(x)\|_{\mathrm{Tr}} \le ct^{-1/2}$, with c independent of x.

Finally, from the above,

$$\operatorname{Trace} D^2g(x) = \xi''(Ix)(DI(x), DI(x)) + \xi'(I(x)) \operatorname{Trace} D^2I(x).$$

But

$$\begin{aligned}\operatorname{Trace} D^2I(x) &= \int_0^\infty h(t) \operatorname{Trace} D^2p_t\, w(x)\,dt \\ &= \int_0^\infty 2h(t)\frac{d}{dt}p_t w(x)\,dt \\ &= 2\left(-h(0)w(x) - \int_0^\infty h'(t)p_t w(x)\,dt\right).\end{aligned}$$

That is,

$$\begin{aligned}\operatorname{Trace} D^2g(x) = {}& \xi''(I(x))(DI(x), DI(x)) \\ & - 2\xi'(I(x))\left[w(x) + \int_0^\infty h'(t)p_t w(x)\,dt\right].\end{aligned}$$

The functions $(DI(x), DI(x))$ and $\int_0^\infty h'(t)p_t w(x)\,dt \in \mathfrak{D}(U)$ by Propositions 3.6, 3.7. Multiplication by the $\mathfrak{D}_0(U)$ functions $\xi'(I(x))$ and $\xi''(I(x))$ gives Trace $D^2g \in \mathfrak{D}_0(U)$.

(5.5) Remark. In the case where B has B-differentiable partitions of unity (for instance, when B is itself a Hilbert space), construction of functions satisfying the conditions in Proposition 5.4 is easier. One chooses a function $g \in D_0^{\infty(B)}(U)$ such that $g = 1$ on W. Since $D^2g(x) \in \mathcal{L}(B, B^*)$, it is trace class, and one uses the representation

$$\operatorname{Trace} D^2g(x) = \int \langle D_B^2 g(x)z, z\rangle p_1(dz)$$

to show that Trace $D^2g \in \mathfrak{D}_0(U)$.

(5.6) Remark. If f if a Lip 1 function with bounded support then Gf satisfies the hypotheses of Proposition 5.2, except that it may not have bounded support and we only know that $\|D^2 Gf(x)\|_{\mathrm{Tr}}$ is uniformly bounded on bounded sets [6].

(5.7) Representation theorem for harmonic local measures. *Let ν be a harmonic local measure on U and f a* Lip 1 *function with properly bounded support in U. Let W be a properly bounded open subset of U such that* Supp f *is properly contained in W. Choose $g \in \mathfrak{D}_0(U)$ as in Lemma* 5.4, *with $g = 1$ on W. Then*

$$\int f(x)\,\nu(dx) = \int \tfrac{1}{2} Gf(x) \operatorname{Trace} D^2 g(x) + (Dg(x), DGf(x))\,\nu(dx).$$

Proof. Let $\phi_f(x) = \frac{1}{2}Gf(x) \operatorname{Trace} D^2 g(x) + (Dg(x), DGf(x))$. Note that $\operatorname{Supp} \phi_f \subset \operatorname{Supp} g$, so the integral on the right is well defined.

We will show that the function $g(x)Gf(x)$ satisfies the hypotheses of Proposition 5.2. Condition (i) is immediate, with

$$D^2(g(x)Gf(x)) = Gf(x)D^2 g(x) + A(x) + g(x)D^2 Gf(x).$$

Here $A(x)$ is the operator defined by

$$(A(x)h, k) = Dg(x)(h)DGf(x)(k) + Dg(x)(k)DGf(x)(h).$$

By Lemma 5.4 and Remark 5.6, the map $x \rightsquigarrow Gf(x)D^2 g(x) + g(x)D^2 Gf(x)$ is defined, bounded and continuous into {trace class operators}. Hence it remains to show this for the map $x \rightsquigarrow A(x)$. But g, Gf are Lip 1, so the proof is exactly the same as in the proof of Lemma 5.4. By Corollary 5.3,

$$\int \operatorname{Trace} D^2[g(x)Gf(x)]\,\nu(dx) = 0.$$

But

$$\begin{aligned}\operatorname{Trace} D^2[g(x)Gf(x)] = {} & (\operatorname{Trace} D^2 g(x))Gf(x) + 2(Dg(x), DGf(x)) \\ & + g(x) \operatorname{Trace} D^2 Gf(x).\end{aligned}$$

The last term $= -2g(x)f(x)$ and $g = 1$ on Supp f. The theorem follows.

Notice that the integrand ϕ_f on the right-hand side of the equation in the previous theorem is supported on a set which is disjoint from the support of f but which "surrounds" it in some sense. That is, a harmonic measure has the property that the integral of smooth functions with respect to the measure is determined by a "boundary value" integral—a rough analog to the mean value property in finite dimensions. Notice also that ϕ_f is basically the Laplacian of the potential function Gf, altered so as to have a specified support. The reason for the elaborate definition of ϕ_f is that even if f is only Lip 1 and bounded, ϕ_f is a test function, as we shall show. The representation of harmonic distributions as measures relies heavily on this property.

(5.8) Lemma. *Under the hypothesis of Theorem* 5.7,

$$\phi_f(x) = \tfrac{1}{2}Gf(x)\,\mathrm{Trace}\,D^2g(x) + (Dg(x), DGf(x)) \in \mathfrak{D}_0(U)$$

If $f_n \to f$ *pointwise and boundedly with* $\bigcup_n \mathrm{Supp}\,(f_n)$ *a properly bounded subset of* W, *then* $\phi_{f_n} \to\to$.

Proof. By construction, g, Trace $D^2g \in \mathfrak{D}_0(U)$ and for $k \geq 1$, Supp $D^kg \subset U - W$ and is a positive distance from both Supp f and U^c. If we show that $Gf \in \mathfrak{D}(U - W)$, then $(Dg(\cdot), DGf(\cdot)) \in \mathfrak{D}_0(U)$ by Proposition 3.7, and $\tfrac{1}{2}Gf(\cdot)$ Trace $D^2g(\cdot)$ also $\in \mathfrak{D}_0(U)$ since Supp Trace $D^2g \subset U - W$.

We first show that for $x \in U - W$, $D^kGf(x) = \int_0^\infty D^kp_tf(x)\,dt$. For each $\bar{h} = (h_1, \ldots, h_k)$, the map $t \rightsquigarrow D^kp_tf(x)(\bar{h})$ is Borel measurable because we can use the equality $p_t\{\|y\| \geq \delta\} = p_1\{\|y\| \geq \delta t^{-1/2}\}$ to write

$$D^kp_tf(x)(\bar{h}) = \int f(x + \sqrt{t}\,z)J_k(\sqrt{t}\,z, \bar{h})\,p_1(dz).$$

We will show the integral converges in W_k norm uniformly for $x \in U - W$.

$$\int_0^\infty |D^kp_tf(x)|_k\,dt \leq \int_0^1 \sup_{U-W} |D^kp_tf(x)|_k\,dt + \int_1^\infty \sup_{U-W} |D^kp_tf(x)|_k\,dt.$$

If $S = \mathrm{Supp}\,f$,

$$\begin{aligned} |D^kp_tf(x)|_k &\leq \sup_{|h_i|=1} \int_{S-x} |f(x+y)|\,|J_k(y, \bar{h})|\,p_t(dy) \\ &\leq \|f\|_\infty (M_kp_t(S - x))^{1/2} t_k^{-1/2}. \end{aligned}$$

By hypothesis, there exists $d > 0$ such that $S - x \subset \{\|y\| \geq d\}$ for $x \in U - W$, giving

$$\sup_{U-W} \sqrt{p_t(S - x)} \leq \left(d^{-2k}\int \|y\|^{2k}p_1(dy)\right)^{1/2} < \infty.$$

Then the first integral is finite and [6, (17)] shows that as $t \to \infty$, $p_t(S - x) = O(t^{-n/2})$ for arbitrary n, uniformly for $x \in U - W$, so the second integral exists. It follows that $\int_0^\infty D^kp_tf(x)dt \in W_k$, with

$$\sup_{U-W}\left|\int_0^\infty D^kp_tf(x)\,dt\right|_k \leq \int_0^\infty \sup_{U-W}|D^kp_tf(x)|_k\,dt < \infty.$$

To prove that this element of W_k is indeed $D^kGf(x)$, one uses induction on k. If true for $k - 1$, then

$$\frac{1}{|h|}\left|D^{k-1}Gf(x+h) - D^{k-1}Gf(x) - \int_0^\infty D^k p_t f(x)(h)\right|_{k-1}$$
$$\leq \int_0^\infty \frac{1}{|h|}|D^{k-1}p_t f(x+h) - D^{k-1}p_t f(x) - D^k p_t f(x)(h)|_{k-1}\,dt.$$

The integrand $\to 0$ as $|h| \to 0$ and is bounded by the integrable function $2\sup_{x \in U-W}|D^k p_t f(x)|_k$.

The argument above shows that

$$|D^k Gf(x) - D^k Gf(y)|_k \leq c\|x-y\| \int_0^\infty \sqrt{p_t\{(S-x) \cup (S-y)\}}\,(t)^{-1/2}dt.$$

But if $x, y \in U - W$, $(S-x) \cup (S-y)$ is again a bounded set contained in $\{\|y\| \geq d\}$, the integral is uniformly bounded, and so the map $x \rightsquigarrow D^k Gf(x)$ is Lip 1 on $U - W$. Hence $Gf \in \mathfrak{D}(U-W)$ and $\phi_f \in \mathfrak{D}_0(U)$.

Finally, if $f_n \to f$ pointwise and boundedly and if $\bigcup_n \operatorname{Supp} f_n$ is a properly bounded subset of W, the preceding estimates show that $\{D^k Gf_n\}$ converges in W norm pointwise and uniformly boundedly on $U - W$. Since $\operatorname{Supp}\phi_{f_n} \subset \operatorname{Supp} Dg$, all n, and

$$D^k\phi_{f_n} = \frac{1}{2}\sum_{\text{finite}} D^i Gf_n(x) D^{k-i}(\operatorname{Trace} D^2 g)(x) + \sum_{\text{finite}} (D^i g(x), D^{k+2-i}Gf_n(x)),$$

we make the obvious estimate on $|D^k\phi_{f_n}|_k$ to prove $\phi_{f_n} \longrightarrow\!\!\!\!\rightarrow$.

Proof of Theorem 2.7. For V a properly bounded open set in U, choose r such that $d(V, U^c) > 3r$. Choose $\mu_t(dx)$ as in Lemma 4.1. Then $T * \mu_t$ is a harmonic local measure in $U_r \supset V$. Let $W = \{x: d(x, V) < r\}$. Choose $g \in \mathfrak{D}_0(U_r)$ as in Lemma 5.4 such that $g = 1$ on W. Then for $f \in \mathfrak{D}_0(U)$ with $\operatorname{Supp} f \subset V$,

$$\int f(x) T * \mu_t(dx) = \int [\tfrac{1}{2}Gf(x)\operatorname{Trace} D^2 g(x) + (DGf(x), Dg(x))]\, T * \mu_t(dx)$$
$$= \int \phi_f T * \mu_t(dx).$$

But since $f, \phi_f \in \mathfrak{D}_0(U_r)$, this can be written

$$T(f * \mu_t) = T(\phi_f * \mu_t).$$

As $t \to 0$, $f * \mu_t \longrightarrow\!\!\!\!\rightarrow f$ and $\phi_f * \mu_t \longrightarrow\!\!\!\!\rightarrow \phi_f$, giving $T(f) = T(\phi_f)$. We will show that there exists a unique measure ν on V such that $T(\phi_f) = \int f(x)\nu(dx)$.

Let $h \in L(V)$ and let $\{h_n\}$ be a sequence of $L_0(V)$ functions which converges boundedly and pointwise to h. Since $\bigcup_n \operatorname{Supp} h_n \subset V$ is properly bounded in W, $\phi_{h_n} \in \mathfrak{D}_0(U)$, $\phi_{h_n} \longrightarrow\!\!\!\!\rightarrow$ and we can define $S(h) = \lim_n T(\phi_{h_n})$. Notice that if $h \in L_0(V)$, then $S(h) = T(\phi_h)$.

S is linear on $L(V)$: this follows from the linearity of T, of differentiation, and of the inner product on H, together with the relationship $G * (ah_1 + bh_2)$

$= aGh_1 + bGh_2$, where $a, b \in \mathbf{R}$. The same argument as in the proof of Theorem 4.3 shows that S satisfies (4.5) and there is a unique regular finite Borel measure ν on V such that for $h \in L(V)$, $S(h) = \int_V h(x)\nu(dx)$. In particular, for the test function f with support in V, $S(f) = T(\phi_f) = T(f) = \int f(x)\nu(dx)$.

The measure ν is independent of V and even of r. Let V_1, V_2 be two properly bounded open subsets of U with the corresponding measures ν_1, ν_2. To show that $\nu_1 = \nu_2$ on $V_1 \cap V_2$, it is sufficient to show $\nu_1(A) = \nu_2(A)$ for A open in $V_1 \cap V_2$. But there is always a sequence of $L_0(V_1 \cap V_2)$ functions which converges to χ_A pointwise and boundedly, so by dominated convergence it is sufficient to show

$$\int h(x)\nu_1(dx) = \int h(x)\nu_2(dx) \quad \text{for } h \in L_0(V_1 \cap V_2).$$

Choose a sequence of test functions f_n with $\operatorname{Supp} f_n \subset V_1 \cap V_2$ and $f_n \uparrow 1$ on $V_1 \cap V_2$. Since h is Lip 1 on U, $f_n(x)p_{1/n}h(x)$ is a test function on U and $f_n(x)p_{1/n}h(x) \to h(x)$ pointwise and boundedly on $V_1 \cap V_2$. Then

$$\begin{aligned}\int h(x)\nu_i(dx) &= \lim_n \int f_n(x)p_{1/n}h(x)\nu_i(dx)\\ &= \lim_n T(f_n(x)p_{1/n}h(x)).\end{aligned}$$

But the last expression is independent of i.

We conclude by showing that the local measure ν which represents the distribution T has smoothness properties in H-directions. We recall that the measures $\{p_t\}$ have the property that for any bounded measurable f, $p_t * f$ is infinitely H-differentiable and this is essentially what we shall show for ν. However, since ν is only defined on properly bounded Borel subsets of U, we must impose conditions on the domain of f and the domain of differentiability of $\nu * f$.

(5.9) Proposition. *If f is a bounded* Lip 1 *function with properly bounded support in $-U = \{x: -x \in U\}$ and $d(\operatorname{Supp} f, -U^c) > r$, and if ν is the local measure representing a harmonic distributiom on U, then $\nu * f$ is defined and infinitely H-differentiable on $B_r(0) = \{x: \|x\| < r\}$.*

Proof. Let $F_x(y) = f(x - y)$. Then for fixed $x \in B_r(0)$, F_x is bounded and Lip 1 with properly bounded support in U.

$$\nu * f(x) = \int f(x-y)\nu(dy) = \int F_x(y)\nu(dy) = T(\phi_{F_x}),$$

where we choose the function g associated with ϕ_{F_x} appropriately (that is, $g \in \mathfrak{D}_0(U)$ such that Supp F_x is properly bounded in $\{g = 1\} = W$),

$$= \int \phi_{F_x}(y)\nu(dy).$$

If $|h|$ is sufficiently small, then $x + h \in B_r(0)$ and Supp F_{x+h} remains properly bounded in W. Hence

$$\nu * f(x + h) = \int \phi_{F_{x+h}}(y)\,\nu(dy).$$

Since F_x and f are Lip 1 with bounded support, GF_x and Gf are H-differentiable away from the supports of F_x, f, with

$$\begin{aligned} D^k GF_x(y) &= \int_0^\infty D^k p_t F_x(y)\,dt \\ &= \int_0^\infty \int_B f(x - y + z) J_k(z, \cdot) p_t(dz)\,dt. \end{aligned}$$

If $h_i \in B^* \subset H$, all $i = 1, \ldots, k$, $J_k(-z, \bar{h}) = (-1)^k J_k(z, \bar{h})$. For such $\bar{h}$, letting $u = -z$,

$$\begin{aligned} D^k GF_x(y)(\bar{h}) &= (-1)^k \int_0^\infty \int_B f(x - y - u) J_k(u, \bar{h}) p_t(du)\,dt \\ &= (-1)^k D^k Gf(x - y)(\bar{h}). \end{aligned}$$

Since B^* is dense in H and both sides are continuous in h_i, we get

$$D^k Gf_x(y) = (-1)^k D^k Gf(x - y).$$

Using the fact that $F_{x+h}(y) = F_x(y - h)$, one has

$$D^k GF_{x+h}(y) = D^k Gf_x(y - h).$$

These equalities, together with estimates similar to those in Lemma 5.8, are used to show that $\nu * f$ is infinitely H-differentiable, with

$$D^k \nu * f(x) = (-1)^{k+1} \int \Big[\tfrac{1}{2} \operatorname{Trace} D^2 g(y) D^k Gf(x - y) - (Dg(y), D^{k+1} Gf(x - y)) \Big]\,\nu(dy).$$

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